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Entanglement of a two-qubit system with anisotropic couplings in nonuniform magnetic fields

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Abstract

We investigate the quantum entanglement of the ground state and the mixed states at finite temperatures for a two-qubit system within the framework of an anisotropic Heisenberg *XYZ* model in the presence of a nonuniform magnetic field. As a measure of the entanglement, the concurrence of the two-qubit states is calculated and is analysed in detail as a function of the coupling constants, magnetic field and temperature. Consequently, we show that the combined influence of the anisotropic interaction and the nonuniformity of the magnetic fields predicts much pronounced entanglement properties.

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1. Introduction

Quantum entanglement is a nonlocal correlation in quantum systems in which the quantum states of two or more objects have to be described with reference to each other, even though the individual objects may be spatially well separated [1]. This strongly correlated phenomenon of the entangled pairs plays a fundamental role in various fields of quantum computation [2] and quantum information such as quantum cryptography [3] and quantum teleportation [4]. These emerging fields stimulate the investigation on quantifying the quantum entanglement and controlling it intensively. Many methods of generating the entanglement have been proposed between the electron spins [5, 6], spin states of quantum dots [7–11], nuclear spins [12], excitons [13], cooper pairs [14] etc. It is therefore of importance to study the entanglement properties of pairs of such qubits. Recently, much attention has been devoted to the interacting Heisenberg spin chain, which is very helpful in gate operations of solid-state quantum computation processors [7, 8, 15].

In spin systems the magnetic field is a useful mean to control the entanglement. The entanglement of the isotropic Heisenberg spin chains has been investigated extensively both in the absence and in the presence of the magnetic field [16–18]. Typically, however, the solid-state heterostructures, which realize qubits, are inhomogeneous and also magnetic imperfections or impurities are likely to be present, giving rise to stray magnetic fields. Further, constructing the identical qubits is still a difficult task, especially in semiconductor technology [19]. Therefore, it is important to study the entanglement of the qubits in the nonuniform magnetic fields. The thermal entanglement of the two-qubit system has been studied in nonuniform magnetic fields, showing that the nonuniformity in the fields affects the entanglement significantly, however, in that the interaction between two qubits was assumed to be isotropic [20].

The isotropic interaction of the spin chain is only an approximation and such an isotropy is usually broken in real systems because, for instance, the spin–orbit coupling may introduce perturbations, among other mechanisms. Accordingly, the investigation of the entanglement when an anisotropic interaction is present constitutes a very interesting and important subject. Previously, an anisotropic *XY* Heisenberg spin chain has been investigated in the absence of magnetic field [18] and also in the presence of uniform magnetic field [21, 22], where it was shown that the entanglement can be manipulated by adjusting the strength of the applied field. The thermal entanglement in the completely anisotropic Heisenberg *XYZ* chain has been investigated and the zero temperature limit was given in [23], however assuming the uniform applied field. It was reported numerically that the entanglement of this model is a monotonously decreasing function of the temperature, when the magnetic fields are absent, for all sets of the coupling constants [24]. In addition, another kind of anisotropic exchange interaction, the Dzyaloshinski–Moriya interaction which arises from the spin–orbit coupling, was considered within the Heisenberg model in studying the thermal entanglement [25]. Most recently, a two-qubit Heisenberg *XXZ* model was treated under an inhomogeneous magnetic field [26], where it was found that the ground-state entanglement is independent of the coupling constant along the field direction.

In this paper, we investigate the effect of both the *nonuniform* magnetic fields and the *anisotropy* of the coupling between two qubits on the entanglement properties. We employ the completely anisotropic Heisenberg *XYZ* spin model to represent the coupled two-qubit systems effectively. Consequently, we have managed to obtain the analytical expression of the concurrence of the system in thermal equilibrium as a function of the coupling constants between the spins, both the homogeneous and inhomogeneous magnetic fields and the temperature. The concurrence is a widely accepted concept of characterizing the degree of the entanglement, which ranges between zero and unity indicating the completely disentangled states and the fully entangled states, respectively. We shall illustrate the various scenarios of how one may manipulate the concurrence from the disentangled value to the fully entangled situation by changing the control parameters. In particular, we focus on the combined influence of the nonuniform fields and the anisotropic interaction on the entanglement, which has not been investigated thoroughly in the literature, thus distinguishing our contribution from other works.

This paper is organized as follows. In section 2 we describe the model spin system considered and obtain its ground states to calculate the concurrences. In section 3 the entanglement properties of the mixed states in thermal equilibrium are presented and analysed in detail, including discussions about the limiting situation to the previous reports. Finally, section 4 contains the concluding remarks.

2. Anisotropic model and the ground state

We use the anisotropic Heisenberg *XYZ* model, representing a two-qubit system effectively, to study the entanglement properties in nonuniform magnetic fields. To this end, we write the

Hamiltonian in the desired form

$$H = \Omega_1\sigma_1 \otimes \sigma_1 + \Omega_2\sigma_2 \otimes \sigma_2 + \Omega_3\sigma_3 \otimes \sigma_3 + (B + b)\sigma_3 \otimes I + (B - b)I \otimes \sigma_3, \tag{1}$$

where σ_j are the Pauli matrices and Ω_j are the interaction coefficients which take care of both the ferro- or anti-ferromagnetic coupling between two spin degrees of freedoms, respectively, where $j = 1, 2, 3$, and, B and b describe the magnetic fields, applied along the z-direction, so that b measures the degree of the inhomogeneity of the applied field at the two spin sites for a given uniform B field. Also, I denotes the identity matrix.

When the temperature is extremely low, the qubit system may be treated as being in the ground state. Here, we first give an account of the ground state properties. Note that both the total spin and its third component are not conserved in our model which contains the nonuniform applied fields and the anisotropic couplings. Consequently, the spin-triplet and -singlet states are not the eigenstates of the system considered. In order to find out the appropriate eigenstates we work in the standard basis for two spins, $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$. After performing a straightforward calculation, we obtain the energy eigenstates of the Hamiltonian, equation (1), as

$$\begin{aligned} |\phi_1\rangle &= \gamma_+|++\rangle + \eta_+|--\rangle, \qquad & |\phi_2\rangle &= \gamma_-|++\rangle + \eta_-|--\rangle, \\ |\phi_3\rangle &= \mu_+|+-\rangle + \nu_+|-+\rangle, \qquad & |\phi_4\rangle &= \mu_-|+-\rangle + \nu_-|-+\rangle, \end{aligned} \tag{2}$$

where the coefficients are defined as

$$\gamma_\pm \equiv \pm\frac{1}{\sqrt{2}}\frac{2B \pm P}{\sqrt{P^2 \pm 2BP}}, \qquad \eta_\pm \equiv \pm\frac{1}{\sqrt{2}}\frac{\Omega_1 - \Omega_2}{\sqrt{P^2 \pm 2BP}}$$
$$\mu_\pm \equiv \pm\frac{1}{\sqrt{2}}\frac{Q \pm 2b}{\sqrt{Q^2 \pm 2bQ}}, \qquad \nu_\pm \equiv \frac{1}{\sqrt{2}}\frac{\Omega_1 + \Omega_2}{\sqrt{Q^2 \pm 2bQ}}.$$

The corresponding energy eigenvalues to these states are given, respectively, as

$$E_1 = \Omega_3 + P, \qquad E_2 = \Omega_3 - P, \qquad E_3 = Q - \Omega_3, \qquad E_4 = -Q - \Omega_3, \tag{3}$$

where definition has been made of

$$P \equiv \sqrt{4B^2 + (\Omega_1 - \Omega_2)^2}, \qquad Q \equiv \sqrt{4b^2 + (\Omega_1 + \Omega_2)^2}.$$

We confirm that equations (2) and (3) reduce to the results reported in [20] when the coupling constants are taken isotropic $\Omega_1 = \Omega_2 = \Omega_3$. Throughout this paper, we shall work in units so that B, Ω_1, Ω_2 and Ω_3 are dimensionless for convenience.

The degree of the entanglement of a quantum state may be specified by the value of what is called the concurrence. We adopt the definition for the concurrence which appears in [27]. In this scheme, one introduces a certain matrix,

$$\mathcal{R} = \rho(\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2), \tag{4}$$

and evaluates the eigenvalues of it, where ρ is the density matrix taken in the aforementioned basis and $*$ indicates the complex conjugate. Then, the concurrence C is defined as

$$\mathcal{C} = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \tag{5}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the positive square roots of the eigenvalues of $\mathcal{R}$ in the descending order. For the pure state, $|\phi\rangle = p|++\rangle + q|+-\rangle + r|-+\rangle + s|--\rangle$, the density operator is given by $\rho = |\phi\rangle\langle\phi|$, and after inserting it into equation (4) one can evaluate the eigenvalues of the matrix $\mathcal{R}$ which determine the concurrence, equation (5), to be

$$\mathcal{C} = 2|ps - qr|. \tag{6}$$

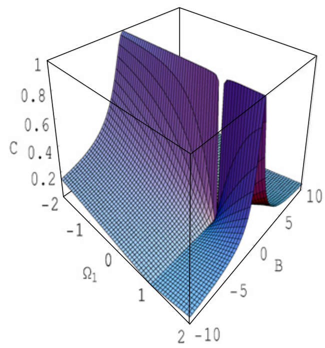

Figure 1. Concurrence of the energy eigenstate $|\phi_2\rangle$ as a function of the uniform field B and the coupling constant Ω_1 where it is assumed that $\Omega_2 = 1$.

The ground state of the system considered is settled by the competition among the various parameters appearing in the Hamiltonian. We find that, when $P > 2\Omega_3 + Q$, the ground state is $|\phi_2\rangle$ and the corresponding energy is E_2. The concurrence of this state is obtained using formula (6) and the result is

$$\mathcal{C}(\phi_2) = \frac{1}{P}|\Omega_1 - \Omega_2|. \tag{7}$$

Note that $\mathcal{C}(\phi_2)$ does not depend on the coupling constant Ω_3 and the inhomogeneity parameter b. The lowest energy state is switched from $|\phi_2\rangle$ to $|\phi_4\rangle$, when we consider the parameters in the opposite range $P < 2\Omega_3 + Q$. In this case, the concurrence of the ground state is obtained as

$$\mathcal{C}(\phi_4) = \frac{1}{Q}|\Omega_1 + \Omega_2| \tag{8}$$

which depends on b instead of B.

In figure 1, we depict the concurrence of the energy eigenstate $|\phi_2\rangle$ which becomes the ground state of the system when the associated parameters obey the inequality $P > 2\Omega_3 + Q$. One can see that the concurrence varies between 0 and 1 as the control parameters are changed. One observes the monotonous decrease of the concurrence with the magnitude of B for the fixed Ω_1 and Ω_2. Also, the sharp drop of the concurrence to zero is seen as a function of Ω_1 at $\Omega_1 = \Omega_2(=1)$, which is clear from equation (7) for $B(\neq 0)$. When $B = 0$, equation (7) shows that $\mathcal{C}(\phi_2) = 1$ identically for $\Omega_1 \neq \Omega_2$. The particular point where $B = 0$ and $\Omega_1 = \Omega_2$ is special in the sense that the concurrence cannot be determined from equation (7). This is because equation (7) is indeterminate when both B and $\Omega_1 - \Omega_2$ vanish simultaneously. We have made an independent calculation for a *XXZ* Heisenberg model with $B = 0$ and found that the corresponding state to the special point is

$$|\phi_2\rangle = |--\rangle,$$

which is disentangled, giving rise to the concurrence of zero.

When the constraint $P = 2\Omega_3 + Q$ is satisfied, the states $|\phi_2\rangle$ and $|\phi_4\rangle$ turn out to be degenerated at the same energy $E_2 = E_4$, and, the ground state is represented as the linear combination of the two states,

$$|\phi_G\rangle = \tfrac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}\alpha}|\phi_2\rangle + |\phi_4\rangle),$$

where α is the relative phase (note that the two states are equally likely to be, according to the principle of the maximum entropy in microcanonical ensemble). When this happens there

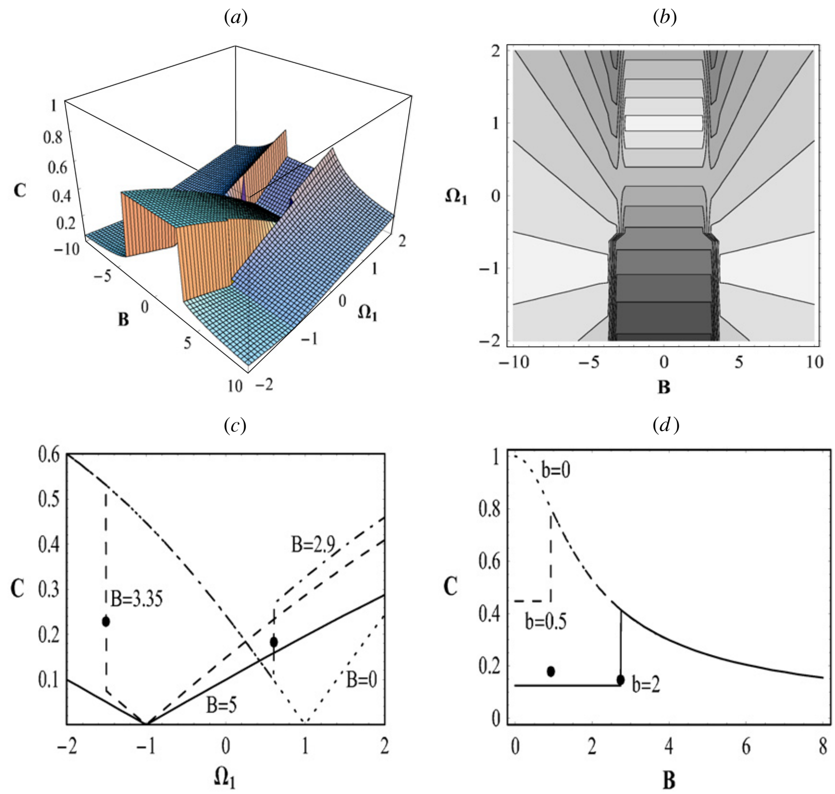

Figure 2. Concurrences of the ground states: (*a*) 3D plot as a function of Ω_1 and B for chosen $\Omega_2 = -1$, $\Omega_3 = 1$ and $b = 2$; (*b*) Contour plot of (*a*); (*c*) $\mathcal{C}$ versus Ω_1 seen along the vertical lines in (*b*) at $B = 0, 2.9, 3.35$ and 5; (*d*) $\mathcal{C}$ versus B seen along the horizontal line in (*b*) at $\Omega_1 = 1.5$ (solid curve) and at two additional values of b.

exists a critical field that divides the ground states of the system between $|\phi_2\rangle$ and $|\phi_4\rangle$ for a given set of the nonuniformity b and the coupling constants, which is determined to be

$$B_c = \left(\Omega_3^2 + \Omega_3\sqrt{4b^2 + (\Omega_1 + \Omega_2)^2} + b^2 + \Omega_1\Omega_2\right)^{1/2}, \tag{9}$$

and, the concurrence of the state $|\phi_G\rangle$ is given as

$$\mathcal{C}(\phi_G) = \frac{1}{2}\frac{|P(\Omega_1 + \Omega_2) - \mathrm{e}^{\mathrm{i}2\alpha} Q(\Omega_1 - \Omega_2)|}{PQ}. \tag{10}$$

(One may consider a critical coupling constant for a given B and other parameters fixed. See figure 2(*c*).)

In figure 2 we manifest the concurrences of the ground states, which is obtained by first judging carefully which state is the ground state for a given set of parameters, Ω_1, Ω_2, Ω_3, B and b, and then by using the right formula for the concurrence. In figure 2(*a*) we present a 3D plot in the different regimes of the coupling constants, where we vary Ω_1 and B continuously for fixed Ω_2 and Ω_3, and for a chosen nonuniformity b (one may vary Ω_2 and/or Ω_3 instead of Ω_1). The corresponding contour plot to figure 2(*a*) is plotted in figure 2(*b*) where, for instance, the effect of the external field can be seen along the B-axis for a chosen Ω_1 and vice versa (note that $C(-B) = \mathcal{C}(B)$). From figures 2(*a*) or (*b*) one notes that there are discontinuous changes in the concurrences at certain B and also Ω_1.

In order to understand the discontinuous behaviour precisely we depict in figure 2(*c*) the concurrence versus Ω_1 by taking four vertical cuts along $B = 0, 2.9, 3.35$ and 5 in figure 2(*b*). Our detailed analysis shows that when $B = 0$ (dotted), the ground state remains in $|\phi_4\rangle$, manifesting a monotonous decrease down to zero at $\Omega_1 = 1$ and then a bouncing back. When $B = 2.9$ (dashed-dotted), the ground state is $|\phi_4\rangle$ until $\Omega_1 < 0.61$ (overlapping with the dotted curve), then it changes abruptly to $|\phi_2\rangle$. Consequently, a discontinuous jump of the concurrence from $\mathcal{C} = 0.1$ to $\mathcal{C} = 0.27$ occurs at $\Omega_1 = 0.61$. Right at this critical Ω_1 two states turn out to be degenerated, and, the corresponding concurrence, $\mathcal{C}(\phi_G) = 0.18$, is indicated as a filled dot, which is calculated from $|\phi_G\rangle$ by choosing $\alpha = 0$ for the illustrative purposes. When $B = 3.35$ (dashed), the ground state is $|\phi_4\rangle$ for $\Omega_1 < -1.5$ (overlapping with the dotted and the dashed-dotted curves) and it undergoes an abrupt change to $|\phi_2\rangle$ for $\Omega_1 > -1.5$. Note that the concurrence vanishes at $\Omega_1 = -1 = \Omega_2$ obeying equation (7). Again, right at the critical $\Omega_1 = -1.5$, the concurrence is given by the degenerated state $|\phi_G\rangle$, whose value is 0.23 marked as a bullet, assuming $\alpha = 0$. When $B = 5$, the ground state remains in $|\phi_2\rangle$, manifested as the solid curve on which a concurrence zero occurs at $\Omega_1 = -1 = \Omega_2$ due to equation (7).

Similarly, in figure 2(*d*) the concurrence is drawn as a function of B at fixed $\Omega_1 = 1.5, \Omega_2 = -1$ and $\Omega_3 = 1$ for three different values of the zero-field splitting parameter, $b = 0, 0.5$ and 2. When $b = 0$ (dotted), the ground state remains in $|\phi_2\rangle$, accordingly the concurrence obeys equation (7), showing the monotonous decrease with B. On the other hand, when the nonuniform field is present, we find the *critical* change of the concurrences with the magnetic field. When we choose $b = 0.5$ (dashed), the ground state is $|\phi_4\rangle$ for $B < 0.93$ and it becomes $|\phi_2\rangle$ abruptly for $B > 0.93$. Similar transition in the ground state occurs, when we use $b = 2$ (solid), around the critical field $B_c = 2.74$. Right at the critical fields, the concurrences are not defined uniquely due to the relative phase α appearing in equation (10). We have chosen $\alpha = 0$ for the illustrative purposes, which specifies $\mathcal{C}(\phi_G) = 0.15$ for the $b = 2$ case. The biggest critical concurrence is $\mathcal{C}(\phi_G) = 0.27$ when $\alpha = \pi/2$ for the same case.

Our work is the generalization of the isotropic two-spin model reported in [20] and of which results are recovered when we set $\Omega_1 = \Omega_2 = \Omega_3 \to J$ in our calculation. The critical field is specified by $B = J + |J|\xi$ and, when this condition holds, the ground state becomes degenerated in the isotropic case as well. We obtain that the concurrence of the degenerated ground state is given by

$$\mathcal{C}(\phi_G) = 1/(2\xi),$$

where

$$\xi \equiv \sqrt{1 + (b/J)^2}.$$

3. Quantum entanglement at finite temperatures

In thermal equilibrium at a temperature T, the density operator of the system is given by $\rho(T) = \mathrm{e}^{-\beta H}/Z$ where $Z = \mathrm{Tr}\,\mathrm{e}^{-\beta H}$ is the partition function and $\beta = 1/(kT)$ with k being the Boltzmann constant. In this case, the state of the system is in a mixture of the ground state and the excited states; accordingly much interesting features are anticipated from the mixing of the various entangled states, depending both on the anisotropy of the coupling and the nonuniformity in the applied field. For the Hamiltonian of the completely anisotropic Heisenberg *XYZ* model in the inhomogeneous magnetic field, given in equation (1), the density operator has the form

$$\rho(T) = \begin{pmatrix} u_- & 0 & 0 & z \\ 0 & v_- & w & 0 \\ 0 & w & v_+ & 0 \\ z & 0 & 0 & u_+ \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned} u_\pm &= \frac{1}{ZP}\,\mathrm{e}^{-\beta\Omega_3}(P\cosh\beta P \pm 2B\sinh\beta P), \\ v_\pm &= \frac{1}{ZQ}\,\mathrm{e}^{\beta\Omega_3}(Q\cosh\beta Q \pm 2b\sinh\beta Q), \\ z &= \frac{\Omega_2-\Omega_1}{ZP}\,\mathrm{e}^{-\beta\Omega_3}\sinh\beta P, \qquad w = -\frac{\Omega_1+\Omega_2}{ZQ}\,\mathrm{e}^{\beta\Omega_3}\sinh\beta Q, \end{aligned} \tag{12}$$

with the partition function Z being

$$Z = 2\,\mathrm{e}^{-\beta\Omega_3}\cosh\beta P + 2\,\mathrm{e}^{\beta\Omega_3}\cosh\beta Q, \tag{13}$$

where P and Q were defined previously. The positive square roots of the four eigenvalues of the matrix $\mathcal{R}$, which is defined in equation (4) and is dependent on temperature T now, are determined to be

$$\begin{aligned} \lambda_I^\pm &= \frac{1}{ZP}\,\mathrm{e}^{-\beta\Omega_3}\left(\sqrt{4B^2+(\Omega_1-\Omega_2)^2\cosh^2\beta P} \pm (\Omega_1-\Omega_2)\sinh\beta P\right), \\ \lambda_{II}^\pm &= \frac{1}{ZQ}\,\mathrm{e}^{\beta\Omega_3}\left(\sqrt{4b^2+(\Omega_1+\Omega_2)^2\cosh^2\beta Q} \pm (\Omega_1+\Omega_2)\sinh\beta Q\right). \end{aligned}$$

The decreasing order of $\lambda_I^\pm$ and $\lambda_{II}^\pm$ depends on the values of the coupling coefficients Ω_j, the parameters B and b of the magnetic field and the temperature T. When the inequality $Q\{4B^2+(\Omega_1-\Omega_2)^2\cosh^2\beta P\}^{1/2} + |\Omega_1-\Omega_2|Q\sinh\beta P > |\Omega_1+\Omega_2|\,\mathrm{e}^{2\beta\Omega_3}P\sinh\beta Q + \mathrm{e}^{2\beta\Omega_3}P\{4b^2+(\Omega_1+\Omega_2)^2\cosh^2\beta Q\}^{1/2}$ holds, we find that the concurrence of the thermal state equation (11) is given by the following analytical formula:

$$\mathcal{C} = \frac{2}{Z}\max\left(0, \frac{1}{P}|\Omega_1-\Omega_2|\,\mathrm{e}^{-\beta\Omega_3}\sinh\beta P - \frac{1}{Q}\,\mathrm{e}^{\beta\Omega_3}\sqrt{4b^2+(\Omega_1+\Omega_2)^2\cosh^2\beta Q}\right). \tag{14}$$

On the other hand, when the sign of the inequality is reversed, the concurrence is determined from the other formula:

$$\mathcal{C} = \frac{2}{Z}\max\left(0, \frac{1}{Q}|\Omega_1+\Omega_2|\,\mathrm{e}^{\beta\Omega_3}\sinh\beta Q - \frac{1}{P}\,\mathrm{e}^{-\beta\Omega_3}\sqrt{4B^2+(\Omega_1-\Omega_2)^2\cosh^2\beta P}\right). \tag{15}$$

Here, we illustrate the concurrences in various situations. In doing so, one has to first decide which formula is to be utilized between equations (14) and (15) in evaluating the concurrence for a given set of parameters. Below, we shall assume that the temperature is in units of the inverse Boltzmann constant, accordingly kT is dimensionless. All other parameters are kept dimensionless as before.

In figure 3 we depict the thermal concurrence as a function of both the temperature and the magnetic field. One can see from the 3D figures, figures 3(*a*) and (*b*), that the concurrence decreases monotonously both with the temperature and the strength of the magnetic field for the isotropic interaction considered, i.e. $\Omega_1 = \Omega_2 = \Omega_3 = 1$. There exist a threshold temperature T_{th} and a critical field B_c for a chosen set of parameters; beyond them the

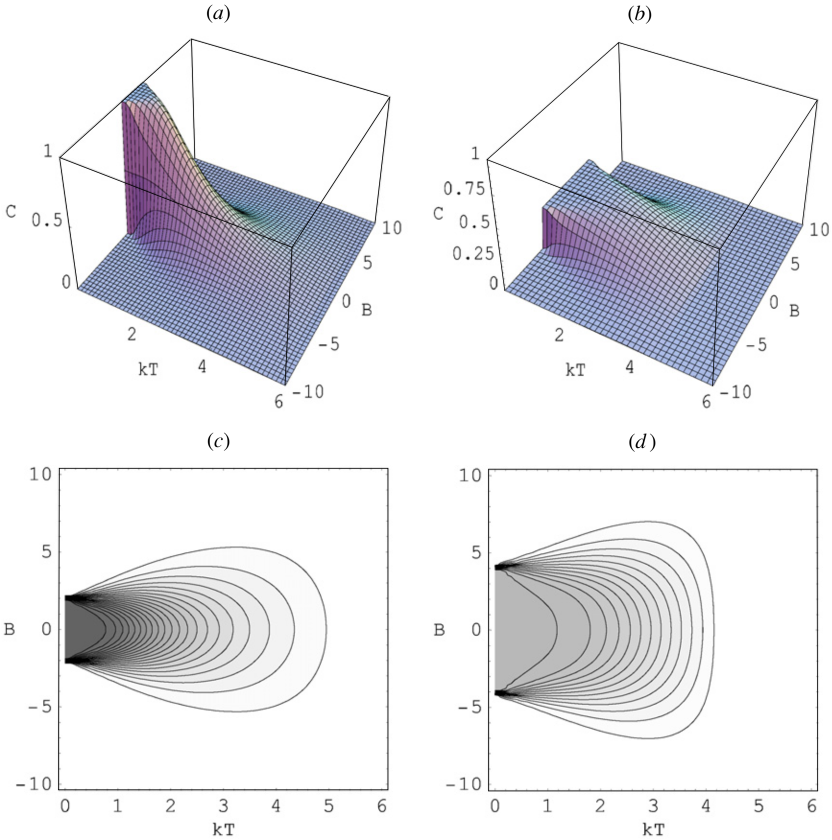

Figure 3. Thermal concurrence versus the temperature T and the magnetic field B: 3D plots (*a*) $b = 0$, (*b*) $b = 3$; contour plots (*c*) $b = 0$, (*d*) $b = 3$, where we have used $\Omega_1 = \Omega_2 = \Omega_3 = 1$.

concurrence gets negligible. This behaviour is seen more clearly in the corresponding contour plots, figures 3(*c*) and (*d*). Also, one notes that the introduction of the nonuniformity in the magnetic field reduces the high-concurrence region significantly.

In figure 4 we illustrate the effect of a completely anisotropic interaction on the concurrences, where $\Omega_1 = 1$, $\Omega_2 = -2$ and $\Omega_3 = 3$ were chosen. Figures 4(*a*) and (*c*) are the case where the magnetic field is uniform, $b = 0$, which are in contrast to figures 4(*b*) and (*d*) where the nonuniformity in the field is introduced, $b = 1$. In general, the figures look similar to those in the isotropic interaction. The role of the nonuniformity in the applied field is again to reduce the high-peak region of the concurrences. However, the details are quite different. A much rich structure shows up and the concurrence no longer decreases monotonously with the temperature as well as the magnetic field, which are analysed in the following figures.

In figure 5 we present the change of the thermal concurrence as a function of temperature. It is seen, in figure 5(*a*), that the concurrence decreases down to zero as the temperature increases for the chosen parameters. Also, one can observe that the threshold temperature can be manipulated with the variation of the anisotropy in the interaction. For the chosen coupling constants T_{th} increases from the left to the right. In figure 5(*b*) one observes the interesting feature of the *revival phenomena* discussed in [23]. We find that for the small nonuniformity the concurrence decreases to the disentangled values as increasing temperature.

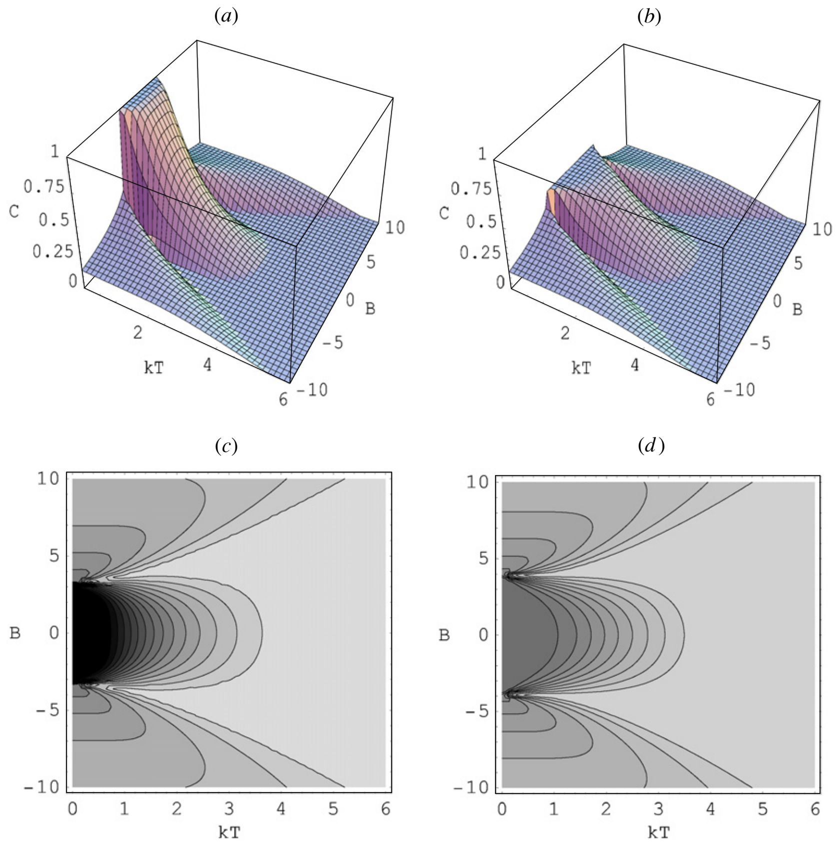

Figure 4. Thermal concurrence versus the temperature T and the magnetic field B: 3D plots (*a*) $b = 0$, (*b*) $b = 1$; contour plots (*c*) $b = 0$, (*d*) $b = 1$, where $\Omega_1 = 1$, $\Omega_2 = -2$ and $\Omega_3 = 3$.

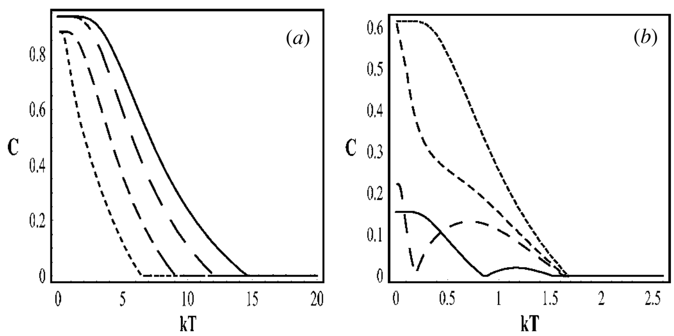

Figure 5. Change of thermal concurrence with temperature: (*a*) for the coupling constants $(\Omega_1, \Omega_2, \Omega_3) = (4, 1.6, 0.2)$, $(4, 1.6, 2)$, $(4, 4, 2)$ and $(4, 4, 4)$ from left to right, while keeping the magnetic fields constant as $B = 2$, $b = 1.5$; (*b*) for several nonuniform fields, $b = 0, 0.9, 1.1$ and 1.6 from top to bottom, where the coupling constants are chosen as $\Omega_1 = 1.5$, $\Omega_2 = -1$ and $\Omega_3 = 1$ and the uniform field is fixed as $B = 1.6$.

On the other hand, as the nonuniformity in the applied field gets bigger, the concurrence shows the oscillatory behaviour. For instance, specifically when $b = 1.1$ in figure 5(*b*), one

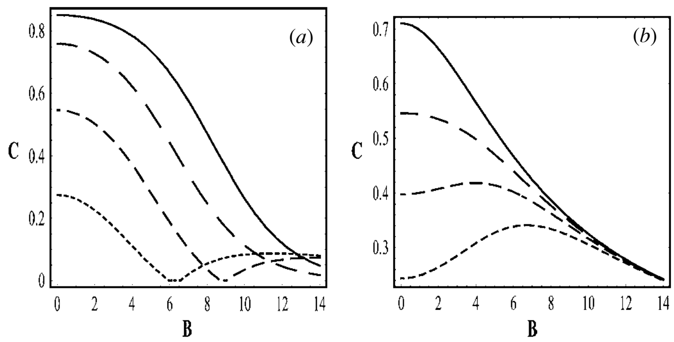

Figure 6. Change of thermal concurrence with magnetic fields: (*a*) for the coupling constants $(\Omega_1, \Omega_2, \Omega_3) = (4, 4, 4), (4, 4, 2), (4, 1.6, 2)$ and $(4, 1.6, 0.2)$, from top to bottom, while fixing $b = 1.5$ and $kT = 4$; (*b*) For several nonuniform fields of $b = 1.0, 4.0, 5.5$ and 7.0, from top to bottom, where the coupling constants are chosen as $(\Omega_1, \Omega_2, \Omega_3) = (4.0, -3.0, -2.0)$ and the temperature is fixed as $kT = 4$.

observes that the concurrence decreases sharply to zero at about $kT = 0.19$, starting from a finite value, and, the concurrence revives as increasing the temperature up to the threshold $kT_{\text{th}} = 1.63$, then it again becomes zero beyond T_{th} and remains so. What happens is as follows: when $kT < 0.19$, the concurrence is given by equation (15) and the value of the second argument in equation (15) is positive, giving rise to the temperature dependence up to $kT = 0.19$. Right at $kT = 0.19$, both the second arguments in equations (14) and (15) turn out to be zero identically, accordingly the concurrence becomes zero. When the temperature is in the range, $0.19 < kT < 1.63$, the concurrence is given by the second argument of equation (14), manifesting the revival of concurrence. Accordingly, the kink appears at $kT = 0.19$. Finally, when $T > T_{\text{th}}$, it is analysed that both the second arguments in equations (14) and (15) are negative; thus the system remains in the completely disentangled states after the threshold temperature. Similar explanation applies to the solid curve in figure 5(*b*) when $b = 1.6$.

In figure 6 we depict the change of the concurrence as a function of the applied field B while the other parameters are fixed. One observes in figure 6(*a*) that the concurrence decreases monotonously for $(\Omega_1, \Omega_2, \Omega_3) = (4, 4, 4)$ and $(4, 4, 2)$. On the other hand, when $(\Omega_1, \Omega_2, \Omega_3) = (4, 1.6, 2)$ and $(4, 1.6, 0.2)$ the revival phenomena appear, which can be understood similarly to our discussion about figure 5(*b*). In figure 6(*b*) it is seen that the concurrence may decrease monotonously or first increase and then decrease with the magnetic field, in this case depending on the zero-field splitting parameter b.

We consider here the situation without the applied fields by setting $B = b = 0$ in the analytical expressions (14) and (15). Consequently, we get that

$$\mathcal{C} = \begin{cases} \max\{0, \mathcal{C}_1\}, \\ \max\{0, \mathcal{C}_2\}, \end{cases} \tag{16}$$

where the first relation holds if $2\Omega_3 < |\Omega_1 - \Omega_2| - |\Omega_1 + \Omega_2|$ and the second relation holds in the opposite domain, and $\mathcal{C}_1$ and $\mathcal{C}_2$ are given as

$$\mathcal{C}_1 = \frac{\sinh\beta|\Omega_1 - \Omega_2| - \mathrm{e}^{2\beta\Omega_3}\cosh\beta(\Omega_1 + \Omega_2)}{\cosh\beta(\Omega_1 - \Omega_2) + \mathrm{e}^{2\beta\Omega_3}\cosh\beta(\Omega_1 + \Omega_2)}, \tag{17}$$

$$\mathcal{C}_2 = \frac{\mathrm{e}^{2\beta\Omega_3}\sinh\beta|\Omega_1 + \Omega_2| - \cosh\beta(\Omega_1 - \Omega_2)}{\cosh\beta(\Omega_1 - \Omega_2) + \mathrm{e}^{2\beta\Omega_3}\cosh\beta(\Omega_1 + \Omega_2)}. \tag{18}$$

It is the thermal concurrence of the two-qubit system with the completely anisotropic interactions in the absence of the inhomogeneous magnetic field. One can check readily that our results coincide with equations (9)–(11) in [24] by replacing $\Omega_1 \to J_x/4$, $\Omega_2 \to J_y/4$ and $\Omega_3 \to J_z/4$, where the monotonous decrease of the entanglement with the temperature was noted numerically. Here we show the change of the concurrence with the temperature analytically. By taking the derivative of equations (17) and (18) with respect to β, we have proved that the following inequalities are satisfied for all sets of the coupling constants:

$$\frac{\partial \mathcal{C}_1}{\partial \beta} > 0 \qquad \text{and} \qquad \frac{\partial \mathcal{C}_2}{\partial \beta} > 0.$$

Thus, we can conclude that

$$\frac{\partial \mathcal{C}}{\partial T} < 0. \tag{19}$$

Also, we discuss another limiting situation of our analytical results. When we take the limit $T \to 0$ and also impose the uniformity of the magnetic fields by setting $b = 0$ in equations (14) and (15), we obtain that

$$\mathcal{C} = \begin{cases} 1, & P < \Omega_1 + \Omega_2 + 2\Omega_3 \\ \frac{1}{2P}(P - \Omega_1 + \Omega_2), & P = \Omega_1 + \Omega_2 + 2\Omega_3 \\ \frac{1}{P}(\Omega_1 - \Omega_2), & P > \Omega_1 + \Omega_2 + 2\Omega_3, \end{cases} \tag{20}$$

where we limit that $\Omega_1 + \Omega_2 > 0$ and $\Omega_1 - \Omega_2 > 0$ in order to compare with other works. This result can be obtained directly from the ground state consideration studied in section 2. Equation (20) is exactly what appears in [23] when replacements are made of $B \to B/2, \Omega_1 \to J_x/2$, $\Omega_2 \to J_y/2$ and $\Omega_3 \to J_z/2$. It shows that the concurrence of the system changes abruptly at a threshold value of the magnetic field, $B_c = \sqrt{\Omega_1\Omega_2 + \Omega_1\Omega_3 + \Omega_2\Omega_3 + \Omega_3^2}$. This critical magnetic field B_c characterizes the quantum phase transition occurring in this system [21, 23].

Finally, it is worthwhile to mention that the concurrences, equation (20), or more generally the ground state concurrences presented in section 2, look peculiarly independent of the coupling constant Ω_3 at a first glance. Similar situation occurs in the Hesenberg XXZ model studied in [26]. However, precisely speaking, this is not the case because the ground state itself is determined by the criteria, $P < 2\Omega_3 + Q$ or $P > 2\Omega_3 + Q$ in the present case, which depends on Ω_3. When the magnetic fields are turned off all together, there is no preferential direction in the system. In this case, the concurrences should be independent of any coupling constants $\Omega_i (i = 1, 2, 3)$, which can be confirmed directly from equations (7) and (8). On the other hand, the Ω_3-dependence is lifted at finite temperatures essentially by the Boltzmann factor, $\exp(-\beta\Omega_3)$, and, the thermal concurrences presented in section 3 manifest this dependence.

4. Summary and conclusion

We have investigated the entanglement properties of the two-qubit system represented effectively by the Heisenberg *XYZ* spin model. We have presented the analytical expressions of the concurrences of the system and their graphical illustrations, taking into account the combined effect from both the anisotropic interaction between two qubits and the nonuniformity in the applied fields.

We have made a through analysis of the ground state and, consequently, have provided the full analytical expression of the concurrences. We have found that the anisotropic interaction between two qubits makes all the energy eigenstates entangled. The system undergoes the critical change between two entangled states as a function of the magnetic field when the nonuniformity is introduced in the applied field. The ground state becomes degenerated at the critical fields, and, the corresponding concurrence to this state takes a definite value which is not in any way connected continuously to either side of the field domain, intriguingly however, it is not defined uniquely due to the relative phase.

At finite temperatures we have also managed to obtain the analytical expression of the Wootters concurrences. We have presented much intriguing features of the thermal concurrences by varying the experimentally controllable parameters. It becomes clear that the concurrence can be manipulated between the highly entangled value and the completely disentangled one in terms of the temperature and the applied field. However, our investigation shows that the details are dependent upon the combined effects of the anisotropy in the interaction between two qubits and the nonuniformity of the applied field. These effects are typically not tractable in real systems, accordingly the transition between an entangled state and a disentangled state may not occur monotonously, but rather in a nontrivial manner. We have manifested the revival phenomenon both as a function of the temperature and the magnetic field.

We have also confirmed that our results recover the existing reports in the limiting situations such as the zero-field limit at finite temperatures and the zero-temperature limit in the presence of the uniform magnetic field, when the interaction between two qubits is anisotropic, also the limit of the isotropic interaction when both the finite temperature and the nonuniform field are present.

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